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Holographic Hydrodynamics and Its Applications in Weyl Semimetal

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Abstract

Holographic Hydrodynamics and Its Applications in Weyl Semimetal

Chiral anomaly is one of the defining properties of Weyl and Dirac fermions.In the introduction chapter, the interest and the reason for studying Weyl and Dirac semimetals in terms of chiral anomaly will be presented along with the reason why and how we use hydrodynamics. In the preliminary chapter, the basics of hydrodynamics, fluid/gravity duality, chiral anomaly will be covered. In the Weyl semimetal chapter, different aspects of Weyl semimetals and also the derivation of chiral anomaly in terms of hydrodynamics will be explored. In the conclusion chapter, review and current challenges in the field of transport phenomenon are presented.

Contents

Chapter 1

Introduction

Chiral anomaly is defined as the quantum term that violates the classical conservation of the chiral current. It was originally observed in neutral pion decay and was initially understood by Adler, Bell, Jackiw in 1969 [13]. Although originating from high energy physics, chiral anomaly continue to hold in some nonrelativistic condensed matter systems[17, 26].

In recent years, topological semimetals have become a major new theme in the field of condensed matter physics. And Weyl semimetal, as the name suggests, is a type of topological semimetal composed of Weyl fermions. It has numerous interesting properties such as its low energy effective band structure has Dirac dispersion, its energy band is topologically protected and it also exhibits discontinuous nontrivial Fermi arc. But most of the above properties are similar to the properties of high temperature superconductor, graphene, topological insulators and quantum hall effect[3, 7, 23, 24]. Out of all the predicted properties of topological semimetals, chiral anomaly still remains interesting due to recent observation in real condensed matter system[17]. Theoretically, there are multiple ways of deriving chiral anomaly[8, 13].

In this thesis, hydrodynamics description of chiral anomaly effects is introduced. Although being one of the oldest,most studied and perhaps hardest branch of classical physics, hydrodynamics has been able to produce fruitful results even in modern physics. In 1941, Landau constructed a quantum theory of superfluid helium [15] which was perhaps the earliest modern application of hydrodynamics. Later on, quantum effects were discovered in hydrodynamics and due to the seminal work[18] by Policastro, Son, Starinets, and Son, the two seemingly unrelated field, string theory and condensed matter start to be connected. In 2002, they showed that according to holography the viscosity/entropy density ratio of the fluid formed from the $\mathcal{N} = 4$ supersymmetric Yang-Mills theory in $3 + 1$ dimensions equals $\eta/s = (1/(4\pi))\hbar/k_B$. And experimental progress made at the same time also showed that quark-gluon plasma as produced in the heavy-ion collisions at the Relativistic Heavy Ion Collider in Brookhaven showed a much less viscous behaviour than expected from perturbative computation. Instead, the plasma seems to behave like ideal fluid. And in 2003/2004 it was also discovered at Relativistic Heavy Ion Collider that the quark-gluon plasma also has a very small η/s value which is very close to the holographic prediction[22, 14]. The mathematical tool, AdS/CFT correspondence is then used in the field of condensed matter physics to study transport phenomenon in 2007[10]. And later on the derivation of effects of chiral anomaly with connection to fluid/gravity correspondence is given by Son and Surowka in 2009[20].

Chapter 2

Preliminary

2.1 Hydrodynamics

We begin with the most general form of the Navier-Stokes equations of motion of a viscous fluid[16]:

$$\rho(\frac{\partial v_i}{\partial t} + v_k \frac{\partial v_i}{\partial x_k}) = -\frac{\partial p}{\partial x_i} + \frac{\partial}{\partial x_k}\eta(\frac{\partial v_i}{\partial x_k} + \frac{\partial v_k}{\partial x_i} - \frac{2}{3}\delta_{ik}\frac{\partial v_l}{\partial x_l}) + \frac{\partial}{\partial x_i}(\zeta\frac{\partial v_l}{\partial x_l}) \tag{2.1}$$

where η is the function of pressure and ζ is the function of temperature. We can also write the above equation in vector form, by making the assumption that the viscosity coefficients η and ζ do not change noticeably throughout the fluid, we have the famous Navier-Stokes equation:

$$\rho[\frac{\partial \boldsymbol{v}}{\partial t} + (v \cdot \nabla)\boldsymbol{v}] = -\nabla p + \eta \Delta \boldsymbol{v} + (\zeta + \frac{1}{3}\eta)\nabla(\nabla \cdot \boldsymbol{v}) \tag{2.2}$$

The reason for introducing Navier-Stokes at the beginning is that any macroscopic system that possesses translational and rotational symmetry with long range interaction at a finite temperature can be modeled in terms of the Navier-Stokes equation. In classical regime, it would seem to be a better idea to use Boltzmann's kinetic equation as a powerful computational tool as long as we are dealing with the collisions of nearly free particles. But as we go into the strongly interacting quantum critical state regime, hydrodynamic equations can still be effective since the conservation laws still holds.

Let us start from the energy-momentum tensor of the simplest ideal fluid which has the form

$$T^{ik} = \begin{pmatrix} e & 0 & 0 & 0 \\ 0 & p & 0 & 0 \\ 0 & 0 & p & 0 \\ 0 & 0 & 0 & p \end{pmatrix} \tag{2.3}$$

It can also be written in terms of the fluid 4-velocity u^i as

$$\boldsymbol{T}^{ik} = w u^i u^k - p g^{ik} \tag{2.4}$$

The energy-momentum itself contains multiple conservation laws of the physical system and the current can be expressed with respect to the 4-velocity u^i as

$$\boldsymbol{j}^i = n\boldsymbol{u}^i \tag{2.5}$$

But in reality, for systems with dissipation, we need to add the extra viscosity and thermal conduction term into the energy-momentum tensor and flux density vector:

$$T_{ik} = -pg_{ik} + wu_iu_k + \tau_{ik} \tag{2.6}$$

$$\boldsymbol{j}_i = n\boldsymbol{u}_i + \boldsymbol{v}_i \tag{2.7}$$

The equations of motion are still contained in the continuity equation and momentum and energy conservation equation.

In order to simplify our problem, we recall some of the most important laws and relations in physics:

$$\partial_\mu S^\mu \geq 0 \tag{2.8}$$

which is the second law of thermodynamics and

$$\mathrm{d}(w - T\sigma)/n = (1/n)\mathrm{d}p - (\sigma/n)\mathrm{d}T \tag{2.9}$$

which is the thermodynamic relation between the relativistic chemical potential. The chemical potential is defined as $\mu = (w - T\sigma)/n$, and w is the enthalpy, i.e. the heat function per unit mass of fluid.

Besides, we also need Landau frame to reduce the complexity brought by the dissipation term:

$$\tau_{ik}u^k = 0 \tag{2.10}$$

$$v_iu^i = 0 \tag{2.11}$$

Now recall the thermal relation derived earlier, and expand the energy-momentum tensor, we have the new form of the energy-momentum tensor of relativistic Navier-Stokes theory as:

$$\begin{aligned} T^{\mu\nu} =& T^{\mu\nu}_{(0)} + T^{\mu\nu}_{(1)}P^{\mu\nu} = g^{\mu\nu} + u^\mu u^\nu T^{\mu\nu}_{(1)} \\ =& -P^{\mu\kappa}P^{\nu\lambda}\left[\eta\left(\partial_\kappa u_\lambda + \partial_\lambda u_\kappa - \frac{2}{d}g_{k\lambda}\partial_\alpha u^\alpha\right) + \zeta g_{k\lambda}\partial_\alpha u^\alpha\right] \end{aligned} \tag{2.12}$$

Similarly, for the current, our new constitutive relation takes the form:

$$J^\mu = nu^\mu - \sigma TP^{\mu\nu}\partial_\nu(\mu/T) \tag{2.13}$$

2.2 Fluid/Gravity Duality

The above relativistic hydrodynamic equations can also be obtained through the calculation in AdS/CFT. The reason is that the Navier-Stokes equation describes the long-range and long-time collective evolution at finite temperature, and the dynamical gravity of the black hole in the bulk described by Einstein gravity can be the "generating functional". This is initially done by Hubeny, Minwalla and Rangamani[12]. By considering the "boosted black branes" which admit AdS_5 solution:

$$ds^2 = -2u_\mu dx^\mu dr - r^2 f(br)u_\mu u_\nu dx^\mu dx^\nu + r^2 P_{\mu\nu}dx^\mu dx^\nu \tag{2.14}$$

And by considering coordinate transformation to Eddington-Finkelstein coordinates, which is necessary to ensure no singularities at the horizon, we have:

$$ds^2 = -2u_\mu dx^\mu dr - r^2 f(r/T)u_\mu u_\nu dx^\mu dx^\nu + r^2\left(u_\mu u_v + \eta_{\mu\nu}\right)dx^\mu dx^\nu \tag{2.15}$$

where natural units are used and AdS radius L is set to be 1. As we change the horizon location as a function of the coordinates x^μ, we also recognize that the necessity of changing other components $u_\mu(x)$ to ensure the equation remains to be a solution to Einstein's equations. We then have:

$$\begin{aligned} ds^2 =& -2u_\mu(x)dx^\mu dr - r^2 f\left(\frac{r}{T(x)}\right) u_\mu(x)u_v(x)dx^\mu dx^\nu \\ &+ r^2\left(u_\mu(x)u_\nu(x) + \eta_{\mu\nu}\right)dx^\mu dx^\nu \end{aligned} \tag{2.16}$$

In order to solve the above metric, we fix the metric:

$$g_{rr} = 0 \tag{2.17}$$

$$g_{r\mu} = -u_\mu \tag{2.18}$$

This give us the constraint equations, which can be simplified in terms of the Landau frame mentioned above. The technique to obtain the energy-momentum tensor will be very straightforward, we apply perturbation expansion relative to the stationary solution and obtain:

$$\begin{aligned} g_{\mu\nu} &= \sum_{n=0}^{\infty} \varepsilon^n g_{\mu\nu}^{(n)}(T(\varepsilon x), u(\varepsilon x)) \\ u^\mu &= \sum_{n=0}^{\infty} \varepsilon^n u^{\mu(n)}(\varepsilon x) \\ T &= \sum_{n=0}^{\infty} \varepsilon^n T^{(n)}(\varepsilon x) \end{aligned} \tag{2.19}$$

$$\begin{aligned} ds^2 =& ds_0^2 + ds_1^2 + \cdots \\ ds_0^2 =& -2u_\mu dx^\mu dr - r^2\tilde{f}(\beta r)u_\mu u_v dx^\mu dx^\nu \\ &+ r^2\left(u_\mu u_v + \eta_{\mu\nu}\right)dx^\mu dx^\nu \\ ds_1^2 =& 2r^2\beta F(\beta r)\sigma_{\mu\nu}dx^\mu dx^\nu + \frac{2}{d} r u_\mu u_v \partial_\alpha u^\alpha dx^\mu dx^\nu \\ &- r u^\alpha \partial_\alpha\left(u_\mu u_\nu\right)dx^\mu dx^\nu \end{aligned} \tag{2.20}$$

Before finally obtaining the Navier-Stokes equation, we recall the Gubser-Klebanov-Polyakov-Witten formula:

$$Z_{\mathrm{QFT}}\left[\phi_0\right] = Z_{\mathrm{QuantumGravity}}\left[\phi \to \phi_0\right] \tag{2.21}$$

which allows us to calculate the expectation value of boundary stress-energy tensor with respect to the boundary metric $^{\mu\nu}$[2, 9]:

$$\begin{aligned} \langle T_{\mu\nu}\rangle &= \frac{2}{\sqrt{-h}}\frac{\delta S_{\mathrm{grav}}}{\delta h^{\mu\nu}} \\ &= \lim_{r\to\infty}\frac{-r^{d+1}}{\kappa^2}\left[K_{\mu\nu} - K h_{\mu\nu} + d h_{\mu\nu} - \frac{1}{d-1}\left({}^h R_{\mu\nu} - \frac{1}{2}{}^h R h_{\mu\nu}\right)\right] \end{aligned} \tag{2.22}$$

By combining the two equation together, we obtain the precise form of relativistic Navier-Stokes equations.

2.3 Chiral Anomaly

Chiral anomaly comes from the fact that, for a massless Dirac Lagrangian which takes the form:

$$\mathcal{L} = i\overline{\psi}\gamma^{\mu}\partial_{\mu}\psi \tag{2.23}$$

there exists a chiral symmetry in classical field theory. But after quantization, this chiral symmetry will be spontaneously broken, the continuity equation and conservation for axial vector current will no longer be 0. Instead, we have:

$$\partial_{\mu}T^{\mu\nu} = F^{\nu\lambda}j_{\lambda} \tag{2.24}$$

$$\partial_{\mu}j^{\mu} = CE^{\mu}B_{\mu} == \frac{e^2}{2\pi^2\hbar^2}E \cdot B \tag{2.25}$$

where the j_5^{μ} is the 4-D axial vector current defined as $j^{\mu 5} = \overline{\psi}\gamma^{\mu}\gamma^5\psi$. It is very unintuitive to understand chiral anomaly at the beginning since there chiral anomaly is purely a quantum mechanical phenomenon with no corresponding classical phenomenon. Here, I will follow a series of Fujikawa's paper[5, 6, 4] to derive the above equation. First, we start from an action for Weyl semimetal:

$$S_w = \int d^4x\overline{\psi}\left(i\gamma^{\mu}\left(\partial_{\mu} + iA_{\mu}\right) - m - b_{\mu}\gamma^{\mu}\gamma^5\right)\psi \tag{2.26}$$

where $\overline{\psi} = \psi^{\dagger}\gamma^0$. This action can written in a partition function as:

$$Z = \int \mathcal{D}\psi\mathcal{D}\overline{\psi}e^{iS_W} \tag{2.27}$$

Now perform a local chiral transformation on ψ and $\overline{psi}$:

$$\psi \to e^{-\frac{i\theta(x)\gamma^5}{2}}\psi \tag{2.28}$$

$$\overline{\psi} \to \overline{\psi}e^{-\frac{i\theta(x)\gamma^5}{2}} \tag{2.29}$$

The partition function Z gives information about phase term and measure term, which also change when we perform the chiral transformation above. We see that the local chiral transformation kills axial vector term and at the end we obtain a massless Dirac action and also the expression for $\theta(x)$:

$$S_D = \int d^4x\overline{\psi}i\gamma^{\mu}\left(\partial_{\mu} + iA_{\mu}\right)\psi \tag{2.30}$$

$$\theta(x) = 2b_{\mu}x^{\mu} = 2\left(\mathbf{b} \cdot \mathbf{r} - \mathbf{b}_0\mathbf{t}\right) \tag{2.31}$$

We would like to know how quantization will influence $j_5^{*\mu}$, from a path integral perspective, we consider the transformation of the measure term:

$$\mathcal{D}\psi'\mathcal{D}\overline{\psi}' \to \mathcal{D}\psi\mathcal{D}\overline{\psi}\,\det\left[e^{i\theta(x)\gamma^5}\right] \equiv \mathcal{D}\psi\mathcal{D}\overline{\psi}e^{i\Delta S_{\theta}} \tag{2.32}$$

We find that:

$$Z' = \int \mathcal{D}\psi\mathcal{D}\overline{\psi}e^{i(S_D + \Delta S_{\theta})} \tag{2.33}$$

where δS_θ and S_D are defined as

$$\Delta S_\theta = \text{Tr}\left[\theta(x)\gamma^5\right], S_D = \int d^4x\overline{\psi}i\gamma^\mu\left(\partial_\mu + iA_\mu\right)\psi = \int d^4x\overline{\psi}i\gamma^\mu D_\mu\psi \tag{2.34}$$

Note that $\{S_D, \Delta S_\theta\} = 0$, which forbids the diagonalization of both terms simultaneously. Now let us assume the the eigenvalue and eigenstate to be ϵ and ϕ, we have:

$$\gamma^\mu D_\mu\phi_n(x) = \varepsilon_n\phi_n(x) \tag{2.35}$$

which satisfies the requirement orthogonality. We plug in this back to the expression for the action ΔS_θ:

$$\Delta S_\theta = \int d^4x\theta(x)\sum_n\phi_n^*(x)\gamma^5\phi_n(x) \tag{2.36}$$

Define the above summation term as:

$$A(x) = \sum_n\phi_n^*(x)\gamma^5\phi_n(x) \tag{2.37}$$

The summation here is complicated since it involves the situation of infinity minus infinity. We use the anti-commutative relation of Dirac gamma matrices$\{\gamma^5, \gamma^\mu\} = 0$, we obtain the new form of (2.31):

$$\gamma^\mu D_\mu\gamma^5\phi_n(x) = -\varepsilon_n\gamma^5\phi_n(x) \tag{2.38}$$

Since eigenstates are orthogonal, only the zero mode eigenstates contribute. We are allowed to find a new set of bases and expand $\mathcal{A}(\S)$:

$$\begin{aligned}\textstyle\int d^4xA(x) &= \textstyle\int d^4x\phi_0^*(x)\gamma^5\phi_0(x)\\ &= \textstyle\int d^4x\sum\phi_{0,\pm}^*(x)(\pm1)\phi_{0,\pm}(x)\\ &= n_+ - n_- = v\end{aligned} \tag{2.39}$$

where

$$\gamma^5\phi_{0,\pm}(x) = \pm\phi_{0,\pm}(x), \quad n_\pm = \int d^4x\phi_{0,\pm}^*(x)\phi_{0,\pm}(x) \tag{2.40}$$

The v denotes the difference of number of left-handed zero mode and right-handed zero mode. It also has a name in math, Pontryagin index of Dirac operator. Now, we use regularization to calculate the summation A(x):

$$\begin{aligned}A(x) &= \lim_{M\to\infty}\sum_n\phi_n^*(x)\gamma^5e^{-\frac{\varepsilon_n^2}{M^2}}\phi_n(x)\\ &= \lim_{M\to\infty}\sum_\pi\phi_n^*(x)\gamma^5e^{-\frac{(\gamma^\mu D_\mu)^2}{M^2}}\phi_n(x)\\ &= \lim_{M\to\infty}Tr\int\frac{a^4k}{(2\pi)^4}e^{-ikx}\gamma^5e^{-\frac{(\gamma^\mu D_\mu)^2}{M^2}}e^{ikx}\end{aligned} \tag{2.41}$$

We put a factor of $e^{-\varepsilon_n^2/M^2}$ into the above integral to ensure the convergence of the summation. We use the anti-commutation relation of Dirac gamma matrices to expand the factor and obtain the final form of $\mathcal{A}(x)$

$$\begin{aligned} A(x) &= = \lim_{M\to\infty} Tr \int \frac{a^4k}{(2\pi)^4} e^{-ikx}\gamma^5 e^{-\frac{\mathcal{D}_\mu \mathcal{D}_\mu \frac{ie}{4}[\gamma^\mu,\gamma^\nu]F_{\mu\nu}}{M^2}} e^{ikx} \\ &== \frac{e^2}{32\pi^2}\varepsilon^{\mu\upsilon\rho\lambda}F_{\mu\upsilon}F_{\rho\lambda} \end{aligned} \tag{2.42}$$

Plug in the expression of A(x) back into ΔS_θ and we find:

$$\Delta S_\theta = \int d^4x\theta(x)A(x) = \frac{e^2}{32\pi^2}\int d^4x\theta(x)\epsilon^{\mu\nu p\lambda}F_{\mu\upsilon}F_{p\lambda} \tag{2.43}$$

Now we are interested in how the chiral anomaly action influence on the current, we calculate the original action after infinitesimal chiral transformation and obtain:

$$S' = \int d^4x\left\{\overline{\psi}i\gamma^\mu D_\mu\psi - \theta(x)/2\left(\partial_\mu j_5^\mu + \frac{e^2}{16\pi^2}\epsilon^{k\upsilon\rho\lambda}F_{\mu\upsilon}F_{\rho\lambda}\right)\right\} \tag{2.44}$$

Set the above equation to be 0 and we obtain the chiral anomaly current equation:

$$\partial^\mu_{\mu j} = -\frac{e^2}{16\pi^2}\epsilon^{\mu\upsilon\rho\lambda}F_{\mu\upsilon}F_{\rho\lambda} \tag{2.45}$$

Now since we are interested in the application of chiral anomaly in condensed matter physics, specifically transport phenomenon, we do the following simplification:

$$\tilde{F}_{\mu\nu} = \frac{1}{2}\epsilon^{\mu\nu\rho\lambda}F_{\rho\lambda} \tag{2.46}$$

$$F_{\mu\upsilon} = \begin{pmatrix} 0 & E_1 & E_2 & E_3 \\ -E_1 & 0 & -B_3 & B_2 \\ -E_2 & B_3 & 0 & -B_1 \\ -E_3 & -B_2 & B_1 & 0 \end{pmatrix} \tag{2.47}$$

$$\overline{F}^{\mu\nu} = \begin{pmatrix} 0 & -B_1 & -B_2 & -B_3 \\ B_1 & 0 & E_3 & -E_2 \\ B_2 & -E_3 & 0 & E_1 \\ B_3 & E_2 & -E_1 & 0 \end{pmatrix} \tag{2.48}$$

$$\epsilon^{\mu\nu\rho\lambda}F_{\mu\nu}F_{\rho\lambda} = -b\mathbf{E}\cdot\mathbf{B} \tag{2.49}$$

Now simply plug in the above equation back to our original expression of the axial current and we find

$$\partial_\mu j_S^\mu = \frac{e^2}{2\pi^2}E\cdot B \tag{2.50}$$

Chapter 3

Weyl Semimetal

3.1 Properties of Weyl Semimetal

The field of Weyl semimetal emerges from the study of the topology of crystalline materials. The theoretical discovery and experimental realization of both Weyl and Dirac semimetals have made this field one of the most important branch of condensed matter physics.

Let us start with the Dirac equation:

$$\left(i\gamma^{\mu}\partial_{\mu} - m\right)\psi = 0 \tag{3.1}$$

with γ being the Dirac gamma matrices. Weyl realized that the Dirac equation can be simplified using the gamma matrices in odd spatial dimensions. For example, in 1+1D dimension, the massless Dirac equation can be written in terms of the eigenstates $\Psi_{\pm}$ as :

$$i\partial_t \psi_{\pm} = \pm p \psi_{\pm} \tag{3.2}$$

which is the 1D Weyl equation. The dispersion relation of this equation is:

$$E_{\pm}(p) = \pm p \tag{3.3}$$

which tells us the chirality of the fermions. Now we extend our analysis to 3D, by denoting

$$\gamma^0 = I \otimes \tau_x \tag{3.4}$$

$$\gamma^i = \sigma^i \otimes i\tau_y \tag{3.5}$$

$$\gamma^5 = -I \otimes \tau_z \tag{3.6}$$

Now we have the massless Dirac equation in 3D

$$\begin{aligned} i\partial_t \psi_{\pm} &= H_{\pm}\psi_{\pm} \\ H_{\pm} &= \mp \vec{p} \cdot \vec{\sigma} \end{aligned} \tag{3.7}$$

The momentum separated pairs given in the above equation are the Weyl semimetals.

Now let us discuss some topological aspects of Weyl semimetals. We know that if a system has time reversal symmetry, we will have the following property of Berry curvature:

$$\Omega_p = \Omega_{-p} \tag{3.8}$$

Also, when we have inversion symmetry, we will have the following property of Berry curvature:

$$\Omega_p = -\Omega_{-p} \tag{3.9}$$

So that means, when we have both the time reversal symmetry $\mathcal{T}$ and the inversion symmetry $\mathcal{P}$, we have the following relation:

$$\Omega_p = 0 \tag{3.10}$$

In Weyl semimetal, doubly degenerate bands arise when both the PT symmetry are satisfied. So under the operation

$$\tilde{\mathcal{T}} = \mathcal{P}\mathcal{T} \tag{3.11}$$

crystal momenta are invariant and thus we have the double degeneracy.

The geometric phase, also known as Berry phase, is a phase difference obtained in the cycle of an adiabatic evolution of Hamiltonian. And Berry connection can be viewed as a local gauge potential associated with the Berry phase. The Berry phase of the Bloch wave functions within a single band n is captured by the line integral of the Berry connection[1]:

$$\mathcal{A}_n(\mathbf{k}) = -i \left\langle u_n(\mathbf{k}) \left| \nabla_{\mathbf{k}} \right| u_n(\mathbf{k}) \right\rangle \tag{3.12}$$

which is equivalently the surface integral of the Berry flux,

$$\mathcal{F}_n^{ab}(\mathbf{k}) = \partial_{k_a} \mathcal{A}_n^b - \partial_{k_b} \mathcal{A}_n^a \tag{3.13}$$

Also, in band theory, we have the net berry flux quantized to integers values since:

$$\int \frac{d^2\mathbf{k}}{2\pi} \mathcal{F}_n(\mathbf{k}) = N_n \tag{3.14}$$

Let us recall the Boltzmann kinetic equation:

$$\frac{df(t,r,p)}{dt} \equiv \frac{\partial f}{\partial t} + r\frac{\partial f}{\partial r} + p\frac{\partial f}{\partial p} = I_c\{f\} \tag{3.15}$$

The corresponding semi-classical equation of motion now takes the form[25]:

$$\begin{aligned} \dot{r} &= \frac{\partial \epsilon_p}{\partial p} + \dot{p} \times \Omega_{p'} \\ \dot{p} &= eE + e\dot{r} \times \boldsymbol{B} \end{aligned} \tag{3.16}$$

The reason for introducing Berry phase here is that with the Berry field, unlike physical magnetic field, we can have magnetic monopoles, which precisely corresponds to the Weyl points in the band structure. Note that one can also obtain chiral anomaly using the Boltzmann kinetic equation, Berry phase and chiral magnetic effect[21]. Through this derivation, one can find that even though in a strongly interacting system, the axial anomalies still exist since they are tying to Fermi surface properties. Besides, in the kinetic equation, we can study anomalies beyond relativistic invariance. And now, I will review chiral anomaly in hydrodynamic regime in the following section.

3.2 Hydrodynamics with Chiral Anomaly

Now as we are equipped with the preliminary and basic properties of Weyl semimetal, we study the chiral anomaly phenomenon in hydrodynamic regime. Let us begin with the current associated with vorticity[20]:

$$J^\mu = nu^\mu - \sigma T \left(g^{\mu\nu} + u^\mu u^\nu\right) \partial_\nu (\mu/T) + \sigma E^\mu + \xi \omega^\mu + \xi_B B^\mu \tag{3.17}$$

with σ being the conductivity, ξ being the new kinetic coefficient, and ω^μ being the vorticity term defined as

$$\omega^\mu = \frac{1}{2}\epsilon^{\mu\nu\lambda\rho} u_\nu \partial_\lambda u_\rho \tag{3.18}$$

In the previous section, we already know the basic form of chiral anomaly. Now we write the chiral anomaly in the following way:

$$\partial_\mu j^\mu = -\frac{1}{8} C \epsilon^{\mu\nu\alpha\beta} F_{\mu\nu} F_{\alpha\beta} \tag{3.19}$$

with C being the anomaly coefficient. We notice that in condensed matter physics scheme, the chiral anomaly is caused by the spectral flow when the gauge vacuum tunnels between different topological configurations. We follow a top-down construction approach in the derivation and we can add a Chern-Simons term in the AdS_5 so that we will obtain the chiral anomaly from the gravity side.

Now, let us start calculate the parity-odd kinetic coefficient ξ by writing out its equation of state:

$$\xi = C\left(\mu^2 - \frac{2}{3}\frac{\mu^3 n}{\epsilon + P}\right) \tag{3.20}$$

where ε and $\mathcal{P}$ are the energy density and pressure. We can modified the above equation in the case of multiple U(1) conserved currents:

$$\xi^a = C^{abc}\mu^b\mu^c - \frac{2}{3} n^a C^{bcd} \frac{\mu^b\mu^c\mu^d}{\epsilon + P} \tag{3.21}$$

Now, we follow quite straightforwardly from the preliminary section 2.1, we use hydrodynamic equations, thermodynamics identity and Landau frame to calculate:

$$\partial_\mu \left(su^\mu - \tfrac{\mu}{T}\nu^\mu\right) = -\tfrac{1}{T}\partial_\mu u_\nu \tau^{\mu\nu} - \nu^\mu\left(\partial_\mu \tfrac{\mu}{T} - \tfrac{E_\mu}{T}\right) - C\tfrac{\mu}{T} E\cdot B \tag{3.22}$$

When C is nonzero, we have anomaly and thus We make the following modification to the U(1) and entropy currents

$$\begin{aligned} \nu^\mu &= -\sigma T P^{\mu\nu}\partial_\nu\left(\tfrac{\mu}{T}\right) + \sigma E^\mu + \xi\omega^\mu + \xi_B B^\mu \\ s^\mu &= su^\mu - \tfrac{\mu}{T}\nu^\mu + D\omega^\mu + D_B B^\mu \end{aligned} \tag{3.23}$$

Follow directly from hydrodynamics, we calculate the hydrodynamic equations:

$$\begin{aligned} \partial_\mu \omega^\mu &= -\frac{2}{\epsilon + P}\omega^\mu\left(\partial_\mu P - nE_\mu\right) \\ \partial_\mu B^\mu &= -2\omega\cdot E + \frac{1}{\epsilon + P}(-B\cdot\partial P + nE\cdot B) \end{aligned} \tag{3.24}$$

Solve the above equations and we will obtain the expression for ξ and ξ_B

$$\xi = C\left(\mu^2 - \frac{2}{3}\frac{n\mu^3}{\epsilon + P}\right), \quad \xi_B = C\left(\mu - \frac{1}{2}\frac{n\mu^2}{\epsilon + P}\right) \tag{3.25}$$

The result here can also be computed using Kubo formula in the calculation of transport coefficients.

Chapter 4

Conclusion

In this thesis, I reviewed the basics of hydrodynamics, fluid/gravity duality, properties of Weyl semimetals, the derivation of chiral anomaly and hydrodynamic modification due to chiral anomaly. The basic theory of Weyl semimetal has been well-established, although there are still some properties that have not been quite understood such as weak anti-localization, negative magnetoresistance and non-saturating magnetoresistance[19, 11]. Weyl semimetal can be seen as an extension of 3D graphene and one of the most fascinating properties of it is the chiral anomaly. The several derivations of chiral anomaly in the context of condensed matter physics have provided great insights not only into the transport phenomenon of Weyl semimetal but also transport phenomenon of other materials and matter. Besides, the mathematical tool brought by AdS/CFT, holographic hydrodynamics has been proven to be very useful in the study of transport phenomenon.

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